

Space-filling Percolation

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A region of two-dimensional space has been filled randomly with large number of growing circular discs allowing only a ‘slight’ overlapping among them just before their growth stop. More specifically, each disc grows from a nucleation center that is selected at a random location within the uncovered region. The growth rate δ is a continuously tunable parameter of the problem which assumes a specific value while a particular pattern of discs is generated. When a growing disc overlaps for the first time with at least another disc, its growth is stopped and is said to be ‘frozen’. In this paper we study the percolation properties of the set of frozen discs. Using numerical simulations we present evidence for the following: (i) The Order Parameter appears to jump discontinuously at a certain critical value of the area coverage; (ii) the width of the window of the area coverage needed to observe a macroscopic jump in the Order Parameter tends to vanish as $\delta \rightarrow 0$ and on the contrary (iii) the cluster size distribution has a power law decaying functional form. While the first two results are the signatures of a discontinuous transition, the third result is indicative of a continuous transition. Therefore we refer this transition as a discontinuous-like continuous transition similar to what has been observed in the recently introduced Achlioptas process of Explosive Percolation. It is also observed that in the limit of $\delta \rightarrow 0$, the critical area coverage at the transition point tends to unity, implying the limiting pattern is space-filling. In this limit, the fractal dimension of the pore space at the percolation point has been estimated to be 1.42(10) and the contact network of the disc assembly is found to be a scale-free network.

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1. INTRODUCTION

In the recently introduced concept of “Explosive Percolation” (EP) it has been suggested that the nature of transition may be discontinuous in some percolation models [1]. This means that the associated Order Parameter, estimated by the size of the largest cluster, should undergo a discontinuous change at the point of transition. In the context of percolation theory such a discontinuous transition can happen only when the largest cluster merges with the maximal of the second largest cluster, which also has a macroscopic size [2, 3]. Though the original model of Explosive Percolation had been studied on Random Graphs [1], later different versions of EP have been studied on the square lattices [4, 5], on scale-free networks [6, 7] and also for real-world networks [8]. Recently it has been shown that, though a class of EP models exhibit very sharp changes in their Order Parameters for finite size systems and appear to exhibit discontinuous transitions, they actually have continuous transitions in the asymptotic limit of large system sizes [9]. Here we propose and study a variant of the Continuum Percolation (CP) model [10] to exhibit a similar discontinuous-like continuous transition.

The original CP model can be stated in the following way. There one finds the minimal density of Lilies, floating at random positions on the water surface of a pond, such that an ant will be able to cross the pond walking on the overlapping Lilies when the radii of the Lilies have a fixed value R [11]. This phenomenon can also be described as how the global connectivity is achieved in a Mobile ad hoc network (MANET) where each node

represents a mobile phone, located at a random position, with a range of transmission R [12]. Depending on R there exists a critical density of Lilies or phones where the long range correlation sets in. It is well-known that in both versions of the CP the transition is continuous and they belong to the same universality class of ordinary lattice percolation [13].

In comparison, here we study the percolation problem in an assembly of growing circular discs. These discs are released one at a time at random positions and they grow at a uniform rate so that at any arbitrary intermediate stage different discs have different radii. In general, a slight overlap among them is allowed when a disc grows to overlap with another one for the first time. This mutual overlap ensures that the global connectivity is achieved at a certain density of discs. In the long time the pattern of discs cover the entire space. We are interested in the study of percolation properties of this space-filling pattern which, to our knowledge, has not been studied yet.

Various models of space-filling patterns have been studied in the literature characterized by their fractal dimensions d_f . In the Apollonian gasket, discs are placed iteratively in the curvilinear triangular spaces between the sets of three mutually touching discs. Consequently the area of the uncovered space gradually vanishes and has the fractal dimension $d_f = 1.305686729(10)$ [14]. In Space-filling bearing patterns a region of two dimensional space is covered by an infinite set of mutually touching discs which can rotate without slipping with a fixed peripheral speed. Different patterns have different fractal dimensions which vary between 1.3057 to 1.4321 [15].

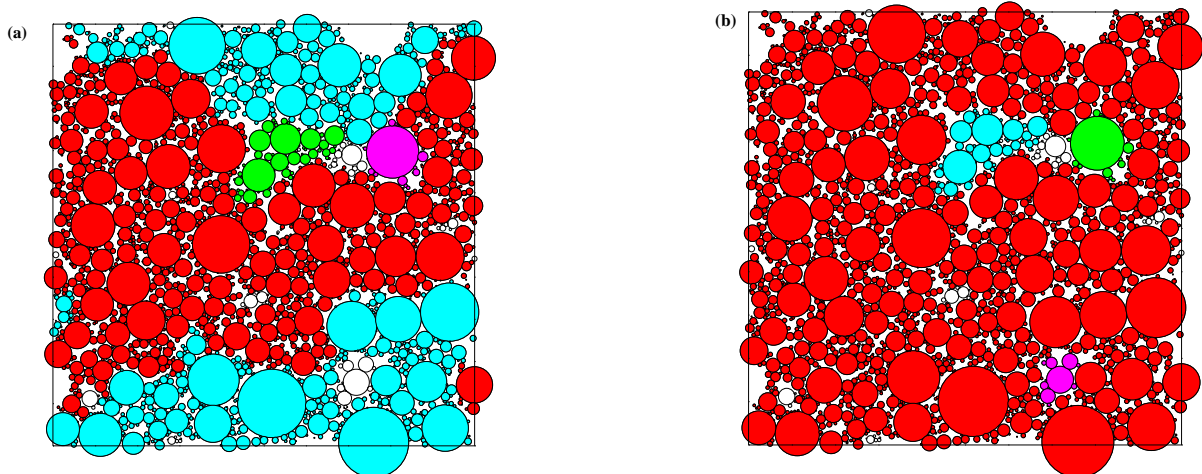


FIG. 1: (Color online) Top four largest clusters of the disc patterns (with $\delta = 0.001$) right before (a) and after (b) the maximal jump in the largest cluster. Different clusters have been shown by circles filled using different colors and rest of the circles are kept unfilled. (a) At time $t = 1973$ the number of discs in the top four large clusters are 1167 (Red); 667 (Cyan); 57 (Green); 16 (Magenta), (b) and at time $t = 1974$ they are 1836 (Red); 57 (Cyan); 16 (Green); 13 (Magenta).

Due to their deterministic algorithms the global connectivity of these patterns is guaranteed. On the other hand, in one model of random space-filling pattern of touching discs, such a global connectivity is not ensured. Here the nucleation centers of the discs are selected at random locations in the uncovered region one after another. After introduction when a disc grows all other discs remain frozen. Such a disc grows till it gets in contact with another disc for the first time when it stops. Such a pattern has the fractal dimension $d_f \approx 1.64$ [16]. Recently space-filling patterns in three dimensional random bearings have been studied in [17]. All these patterns, in the limit of infinite number of generations, are space-filling. It is known that, while this limit is being taken, the dust of remaining uncovered pore spaces form a Fractal set. The fractal dimension d_f can be estimated following [18].

In section 2 we describe the model in detail, the data and their analysis are presented in section 3. Section 4 describes the associated contact network and finally we conclude in section 5.

2. THE MODEL

The pattern of circular discs is generated within a unit square box placed on the $x - y$ plane. The radii of all discs grow uniformly using a continuously tunable rate δ . To generate a pattern, a specific value of δ is assigned for all discs. The time t is a discrete integer variable that counts the number of discs released. Therefore at time t the pattern has exactly t discs of many different radii. Initially the square box is completely empty. Then at each time step a new disc with zero radius is introduced. Within the square box the assembly of discs cover a re-

gion of space which is called the ‘covered region’, the remaining space not covered by any of the discs is referred as the ‘uncovered region’. While growing, once a disc overlaps with another disc, it stops immediately and does not grow any further. Such a disc is called a ‘frozen’ disc.

At any arbitrary intermediate time step the following activities take place: (i) A point is randomly selected anywhere within the uncovered region. A circular disc of radius δ is placed with its center fixed at this point. (ii) Simultaneously, the radii of all other non-frozen discs are also increased synchronously by the same amount δ . Every growing disc is checked if it has overlapped with any other disc, if so, it’s growth is stopped and is declared as a frozen disc.

Gradually, the space within the square box is increasingly covered by the discs and therefore the amount of uncovered area decreases with time. We define the control variable p as the sum of the areas of all discs. It may be noted that p is slightly larger than the actual “area coverage” since the overlapped areas are doubly counted in the total sum of disc areas. However it has been observed that the total overlap area tends to vanish in the limit of $\delta \rightarrow 0$ and we would refer p as the area coverage in the following discussion.

In a particular run, the simulation is stopped only when the area coverage p reaches a pre-assigned value or some pre-defined condition becomes valid. For example, to reach the percolation point, the run is terminated only when a global connectivity appears through the overlapping discs from the top to the bottom for the first time. If one continues further, a stage would come when the different pieces of uncovered regions would be so tiny that any newly introduced disc would freeze immediately at

the first time step. This would be the natural exit point of the simulation. However in most cases of our calculation simulations were run up to the percolation point.

We are interested to study the percolation process of this growth model. Multiple overlapping discs form different clusters. Discs of a specific cluster are connected among themselves through overlaps. Size s of a cluster is determined by the sum of the areas of all discs of the cluster. It has been observed, that in an arbitrary pattern near the percolation point, typically there are two large clusters. For example in Fig. 1(a) we exhibit a typical disc pattern just before the percolation point at time $t = 1973$ grown at a rate $\delta = 0.001$. Four top largest clusters are shown by discs, filled using different colors. Right at the next time step $t = 1974$ two small discs connect the top two largest clusters so that the size of the largest cluster jumps from 0.439 to 0.747 (Fig. 1(b)). This behavior is typical of the percolation process studied here. Prior to the percolation point the largest and second largest clusters have a tendency to compete and grow simultaneously while maintaining their comparable sizes. We define the percolation point when the Order Parameter jumps by a maximum amount in a single time step. This happens only when the largest cluster merges with the maximal of the second largest cluster. In the following we present simulation results exhibiting this behavior.

Given a specific growth rate δ one generates the disc assembly till the percolation point. At this stage a probability distribution $n(r, \delta)$ of radii r of the discs in the pattern is defined. Consequently, for the subset of discs whose radii are at least r , one further estimates the cumulative distribution $N(r, \delta)$; the total perimeter $P(r, \delta)$ of all discs in the subset and $A(r, \delta)$ as the total remaining uncovered area external to all discs in the subset. It is assumed that in the limit of $r \rightarrow 0$ all these quantities vary as some powers of r as follows [18]:

$$\begin{aligned} n(r, \delta) &= \sum_{r_i=r} 1 \sim r^{-(d_f+1)} \\ N(r, \delta) &= \sum_{r_i \geq r} 1 \sim r^{-d_f} \\ P(r, \delta) &= 2\pi \sum_{r_i \geq r} r_i \sim r^{1-d_f} \\ A(r, \delta) &= 1 - \pi \sum_{r_i \geq r} r_i^2 \sim r^{2-d_f} \end{aligned} \quad (1)$$

3. THE RESULTS

Let $s_m(p, \delta)$ denote the size (i.e., maximal covered area) of the largest cluster. Then the Order Parameter $\mathcal{C}(p, \delta)$ of the growth process is determined by the average size of the largest cluster of the pattern for an area coverage p ,

$$\mathcal{C}(p, \delta) = \langle s_m(p, \delta) \rangle \quad (2)$$

the average being taken over a large number of uncorrelated growth processes. Since no checking of the global

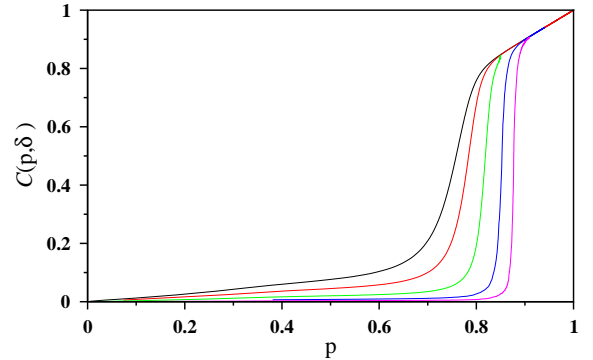


FIG. 2: (Color online) Order Parameter $\mathcal{C}(p, \delta)$ has been plotted against the area coverage p for growth rates $\delta = 0.002$ (black), 0.0008 (red), 0.0002 (green), 0.00004 (blue) and 0.00001 (magenta).

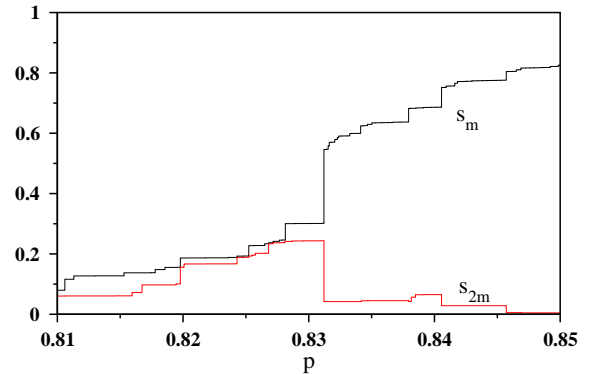


FIG. 3: (Color online) Variation of the sizes of the largest (black) cluster s_m^α and the second largest (red) cluster s_{2m}^α with the area coverage p for a single run and for $\delta = 0.0001$. While s_m^α increases monotonically, s_{2m}^α increases to a maximum and then drops to a small value when the second largest cluster merges with the largest cluster. This corresponds to the maximal jump in the largest cluster and is identified as the percolation point for the α -th run.

connectivity is required in this part of the simulation, we have used the periodic boundary condition along both the x and y axes. In Fig. 2 we plotted $\mathcal{C}(p, \delta)$ with p for five different values of the growth rate δ . It has been observed that for every plot that around a specific value of $p = p_c(\delta)$ the growth of the Order Parameter is very sharp. This happens because for a typical run α the maximal jump in $s_m^\alpha(p, \delta)$ takes place at p_c^α .

A closer look into the growth process reveals that this maximal jump in the largest cluster occurs only when the maximum of the second largest cluster merges with the largest cluster. This has been exhibited explicitly in Fig. 3 where we plot the sizes of the largest cluster s_m^α and that of the second largest cluster s_{2m}^α with the area coverage p for a single run α . While s_m^α grows monotonically, growth

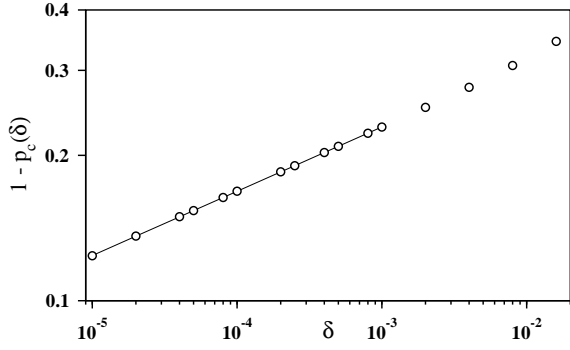


FIG. 4: The deviation $1 - p_c(\delta)$ of the percolation threshold from unity has been plotted with the growth rate δ on a log-log scale. It is observed that as $\delta \rightarrow 0$ the deviation vanishes as a power of δ as: $1 - p_c(\delta) \sim \delta^{0.133}$.

of s_{2m}^α is not so because it reaches to a maximum and then falls to a much lower value. We assume a nomenclature that always the smaller cluster merges with the larger cluster. Therefore, when s_{2m}^α merges with s_m^α it is the third largest cluster s_{3m}^α that becomes the second largest cluster s_{2m}^α . This may happen a few times and we mark that particular value of $p = p_c^\alpha$ where the maximal jump in s_m^α takes place.

We define $p_c^\alpha(\delta)$ as the percolation threshold of the α -th run [3]. This value is then averaged over a large number of un-correlated runs and the percolation threshold $p_c(\delta)$ is defined as:

$$p_c(\delta) = \langle p_c^\alpha(\delta) \rangle. \quad (3)$$

It may be observed in Fig. 2 that the values of percolation thresholds $p_c(\delta)$ are gradually shifting towards unity as $\delta \rightarrow 0$. To see this approach quantitatively, we plotted $1 - p_c(\delta)$ against δ on a log-log scale in Fig. 4. For small values of δ the data points indeed fit nicely to a straight line with slope 0.133. Therefore we may write that as $\delta \rightarrow 0$,

$$1 - p_c(\delta) \sim \delta^{0.133}. \quad (4)$$

This implies that in the limit of $\delta \rightarrow 0$ the area coverage at the percolation point becomes unity. In other words when the growth rate is infinitely slow, the global connectivity would appear for the first time when the entire space would be covered by the discs and this limiting pattern is therefore space-filling. We therefore call this problem as “Space-filling percolation”.

The percolation threshold has also been determined using the usual definition, i.e., when the global connectivity appears for the first time in the system. In this case, periodic boundary condition along the x -axis and open boundary condition along the y -axis have been used. For a specific run, as more and more discs are released, we keep track if the connectivity between the top and the

bottom boundaries of the unit square box through the system of overlapping discs has appeared. When such a connectivity appears for the first time, we refer the corresponding pattern of discs as the percolation configuration and define the total area coverage as the second definition of the percolation threshold for this particular run. Like before, an average of these threshold values for a large number of independent runs gives us the value of $p_c(\delta)$. It has been observed that the percolation thresholds measured using two methods differ by small amounts, e.g., 0.039 and 0.019 for $\delta = 0.001$ and 0.0001 respectively and this difference gradually diminishes as $\delta \rightarrow 0$.

At the percolation threshold $p_c^\alpha(\delta)$ of the α -th run the disc pattern has one or more discs which have the largest radius $r_m^\alpha(\delta)$. The radius $\langle r_m(\delta) \rangle$ of the largest disc, averaged over many runs, decreases as the growth rate δ decreases. A power law form along with a logarithmic correction has been observed for this variation:

$$\langle r_m(\delta) \rangle \sim \{\delta \log(1/\delta)\}^{0.331}. \quad (5)$$

In a similar way the average radius of a disc at the percolation point also has a power law variation as in the following.

$$\langle r(\delta) \rangle \sim \delta^{0.666}. \quad (6)$$

It may be noted that though $\langle r(\delta) \rangle < \langle r_m(\delta) \rangle$ the exponent of the former is larger since as $\delta \rightarrow 0$ the value of $\langle r(\delta) \rangle$ decreases much faster than $\langle r_m(\delta) \rangle$.

Eqn. (4) may be compared with the well-known relation of percolation theory that connects the ordinary site or bond percolation thresholds $p_c(L)$ for finite size systems of length L with the correlation length exponent ν as: $p_c(\infty) - p_c(L) \sim L^{-1/\nu}$. In this case we may consider that the characteristic size of the objects filling the space is given by the average $\langle r_m(\delta) \rangle$ of the maximal radius of the discs. Therefore for a given growth rate δ the system size (equivalent to L), may be measured using $\langle r_m(\delta) \rangle$ as the yardstick and therefore $L \sim 1/[\langle r_m(\delta) \rangle]$. Using Eqn. (4) and Eqn. (5) we can write:

$$1 - p_c(\delta) \sim \delta^{0.133} \sim [\langle r_m(\delta) \rangle]^{0.133/0.331} \sim L^{-0.402}. \quad (7)$$

Using this equation we estimate $\nu = 0.331/0.133 \approx 2.49$ which is compared with $\nu = 4/3$ for ordinary percolation. We conjecture that in our case ν may be exactly equal to 5/2.

Let us denote that the maximal jump in the Order Parameter by $\Delta_m \mathcal{C}(\delta)$. This is the average of the maximal jumps in the size of the largest clusters over a large number of independent runs, i.e., $\Delta_m \mathcal{C}(\delta) = \langle \Delta_m s_m(p, \delta) \rangle$. In Fig. 5 we plot $\Delta_m \mathcal{C}(\delta) - \Delta_m \mathcal{C}(0)$ with δ on a log-log scale. Since $\Delta_m \mathcal{C}(0)$ cannot be estimated directly we tried with different trial values of $\Delta_m \mathcal{C}(0)$ to make the plot which fits best to a straight line. Though there is an

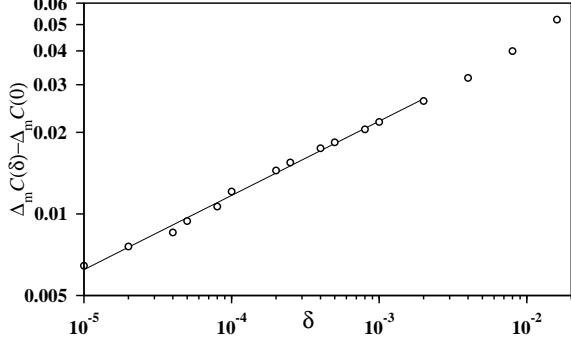


FIG. 5: How the average value of the maximal jump in the Order Parameter $\Delta_m C(\delta)$ approaches to $\Delta_m C(0)$ has been shown. After some initial curvature the plot fits to a straight line as $\delta \rightarrow 0$. This implies that Eqn. (8) indeed holds good with $\Delta_m C(0) = 0.16$ and $\mu = 0.274$.

initial curvature for large values of δ , the latter points obtained as $\delta \rightarrow 0$ fit nicely to a straight line. This implies that the variation can be termed as a power law like:

$$\Delta_m C(\delta) = \Delta_m C(0) + A\delta^\mu \quad (8)$$

with $\Delta_m C(0) = 0.16$, $A = 0.15$ and $\mu = 0.274$. This relation can be interpreted that even in the limit of $\delta \rightarrow 0$ the average maximal jump in the Order Parameter i.e., the area coverage of the largest cluster, is a finite fraction of the entire area of the disc pattern. Therefore this is also another signature of the discontinuous percolation transition in our model.

The rapidity with which the Order Parameter increases in Fig. 2 at the percolation threshold can also be quantified by measuring the width of the window around the percolation threshold following the method used in [1]. For a single run, we define $p_{1/10}$ as the minimum value of the area coverage p for which $C > 1/10$. Similarly $p_{1/2}$ is the minimum value of p for which $C > 1/2$. The difference in these two area coverages $p_{1/2} - p_{1/10}$ is the size of the window through which a 40 percent jump in the Order Parameter takes place.

Averaging over a large number of uncorrelated runs we estimated $\langle p_{1/10} \rangle$, $\langle p_{1/2} \rangle$ and $\langle p_{1/2} - p_{1/10} \rangle$. In Fig. 6(a) we plotted $1 - \langle p_{1/10} \rangle$ and $1 - \langle p_{1/2} \rangle$. using a log-log scale. It is observed that $1 - \langle p_{1/10} \rangle$ has some initial curvature for large values of δ but as $\delta \rightarrow 0$ the curve become straight as a power law: $1 - \langle p_{1/10} \rangle \sim \delta^{0.166}$. On the other hand $1 - \langle p_{1/2} \rangle$ fits to a nice straight line over the entire range of δ implying a power law variation $1 - \langle p_{1/2} \rangle \sim \delta^{0.128}$. In Fig. 6(b) we plotted $\langle p_{1/2} - p_{1/10} \rangle$ against δ . Here also, apart from some initial curvature for large δ the curve fits to a power law

$$\langle p_{1/2} - p_{1/10} \rangle \sim \delta^{0.50}. \quad (9)$$

Therefore as $\delta \rightarrow 0$ a 40 percent increase in the Order Parameter requires a vanishingly small change in the area

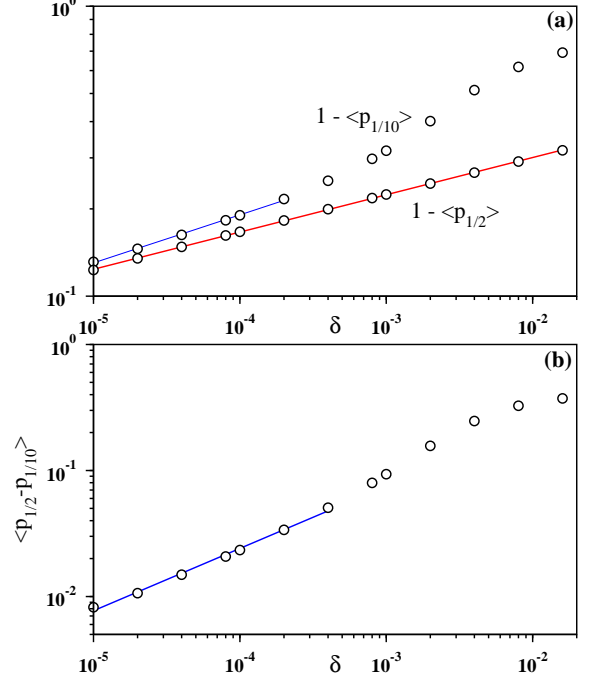


FIG. 6: (Color online) (a) Both $1 - \langle p_{1/10} \rangle$ and $1 - \langle p_{1/2} \rangle$ tends to vanish as $\delta^{0.166}$ and $\delta^{0.128}$ respectively when $\delta \rightarrow 0$. (b) Across the window $\Delta p = p_{1/2} - p_{1/10}$ the Order Parameter $C(p, \delta)$ jumps from $1/10$ to $1/2$. The average size of this window has been plotted with δ on log-log scale and is observed to vanish as $\delta^{0.50}$ as $\delta \rightarrow 0$.

coverage. This is again another evidence that the percolation transition in this model is likely to be a discontinuous transition.

The percolation transition in our model is analyzed by yet another method, this time studying the cluster size distribution at the percolation point. For this study we defined the cluster size S in a different way, this time it is the number of discs belonging to a specific cluster. In Fig. 7 we have presented the binned data for the probability distributions of the cluster sizes for four different values of δ . The cluster size distribution data have been collected only when the global connectivity appears for the first time. All four $D(S, \delta)$ vs. S plots on the log-log scale have similar nature. After some initial slow variation, the $\log D(S, \delta)$ decreases linearly with $\log S$ in the intermediate power law regime. Finally at the tail of the distribution there is a hump, meaning an enhanced probability for the large clusters which connects the two ends of the system. It is assumed and which seems to be very likely that as δ decreases the position of the hump shifts systematically to large values of S and therefore in the limit of $\delta \rightarrow 0$ the entire intermediate regime would fit to a power law of the form: $D(S) \sim S^{-\tau}$. We conclude an average value of $\tau = 2.14(2)$ which is to be compared with $\tau = 187/91$ for ordinary percolation [13]. This power law

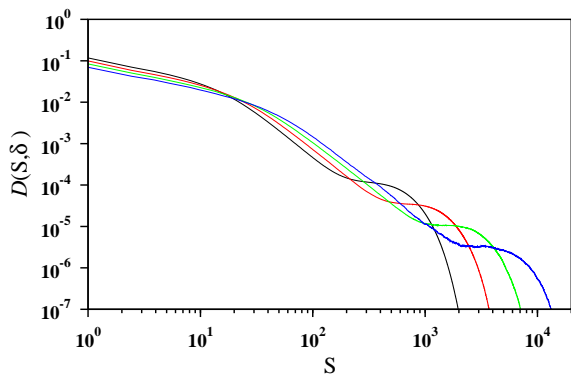


FIG. 7: (Color online) Display of the binned data for the normalized cluster size distribution $D(S)$ measured by the number of discs S in a cluster. From the slopes in the intermediate regime we obtained $\tau = 2.16, 2.13, 2.14$ and 2.13 respectively for growth rates $\delta = 0.002$ (black), 0.001 (red), 0.0005 (green) and 0.00025 (blue), δ decreases from left to right. The intermediate power law regime gets elongated as δ decreases and we conclude a value of the associated exponent $\tau = 2.14(2)$.

distribution of the cluster sizes can be interpreted as an indication of a continuous transition for the percolation transition, in contradiction to the discontinuous transition concluded from Eqns. (8) and (9).

Here we recall that the original model of Explosive Percolation, which goes by the name of ‘Achlioptas Process’ [1] has a similar story. For this model, most of the numerical results indicated that the associated percolation transition is discontinuous. However, recently Riordan and Warnke have rigorously proved that a class of models using Achlioptas type processes are in fact continuous in the asymptotic limit of large size graphs [9]. We conclude that the percolation transition in our model also behaves similarly to the Achlioptas Process, so that though apparently it exhibits the behavior alike to a discontinuous transition, it is indeed a continuous transition.

Finally we measured the fractal dimension of the dust of pore spaces right at the percolation threshold using the Eqn. (1). We considered a large sample of uncorrelated disc patterns that have been grown at a rate δ . A periodic boundary condition has been imposed along the x direction and the pattern is grown till a global connection appears along the y axis when further growth is terminated. For each pattern we estimated the following quantities: the distribution $n(r, \delta)$ of the number of discs of radii r , its cumulative distribution $N(r, \delta)$, total perimeter $P(r, \delta)$ and the remaining uncovered area $A(r, \delta)$. We plot all four quantities in Fig. 8 using a log – log scale for $\delta = 0.000125$. For each curve the scaling appeared in the intermediate regime of disc radii. Estimation of slopes of these curves in their scaling regions and using Eqn. (1) we have obtained the values of the fractal dimensions as 1.42, 1.41, 1.40 and 1.46 respec-

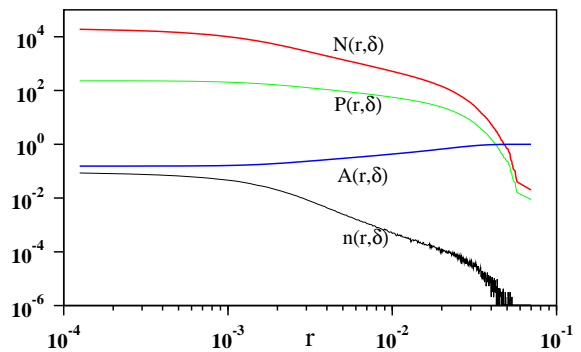


FIG. 8: (Color online) The probability distribution $n(r, \delta)$ of the number of discs of radii r , its cumulative distribution $N(r, \delta)$, total perimeter $P(r, \delta)$ and the remaining uncovered area $A(r, \delta)$ for disc patterns grown at a rate $\delta = 0.000125$. The slopes of each curves gives an estimate of the fractal dimension d_f and an average value of $d_f = 1.42(10)$ has been obtained.

tively. Clearly a large scatter of the estimated value of d_f is present, yet this data indicates that the d_f is like to be around 1.42 with rather large error of around 0.10. A more accurate estimation needs patterns to be generated using even smaller value of the growth rate δ .

4. CONTACT NETWORK

A contact network for the assembly of overlapping discs may be defined identifying the centers of the discs as nodes. In addition a link between a pair of nodes is introduced if and only if their corresponding discs overlap [19]. As time passes the contact network grows in the number of nodes as well as links. In Fig. 1(a) we have exhibited the disc pattern at the percolation threshold where different discs are of different radii. In general the large discs have overlaps with many other discs and therefore in the contact network these nodes form the hubs of the network. In the same way smaller discs have fewer links but their numbers are more. The contact network corresponding to the Fig. 1(a) has been exhibited in Fig. 9. Since there are many clusters, the network is not a singly connected graph. The four top large clusters are represented by four sub-graphs of the network.

The degree k_j of a node is the number of links meeting at node j . Here that would be equal to the number of other k_j discs that have overlap with the j -th disc. Typically such networks are scale-free networks which have power law degree distributions. Similarly we expect that for the contact network of the disc pattern, the largest component of the graph, right at the percolation threshold has the degree distribution $D(k) \sim k^{-\gamma}$, in the limit of $\delta \rightarrow 0$ where γ is the degree distribution exponent to be estimated. For finite δ we have calculated the prob-

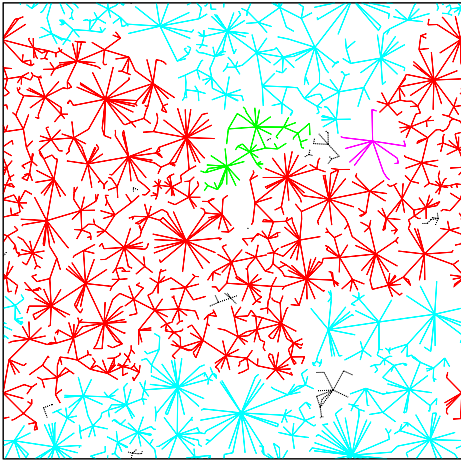


FIG. 9: (Color online) The picture of the contact network corresponding to the disc pattern in Fig. 1(a). A link between a pair of discs that overlapped is drawn by a straight line joining their centers. The pattern has many clusters. The sub-graphs corresponding to the top four largest clusters are displayed by the same colors as used in Fig. 1(a).

ability distribution $D(k, \delta)$ which is the probability that a randomly selected node has degree k . In Fig. 10(a) we plot $D(k, \delta)$ against k on a log-log scale for the three different values of δ . Apart from the very small and large values of k the curves are quite straight in the intermediate regimes indicating that the degree distributions indeed have power law variations in the intermediate range of degree values. A direct measurement of slopes gives the values of $\gamma(\delta)$ actually depend on δ and extrapolate like $\gamma(\delta) = \gamma(0) - 5.8\delta^{0.39}$ with the extrapolated value of $\gamma(0) = 2.80$. This analysis is supported by a finite size scaling analysis of the same data. In Fig. 10(b) we plot the same data used in Fig. 10(a) but scale both the axes with suitable powers of δ . The best data collapse corresponds to

$$D(k, \delta)/\delta^\eta \sim \mathcal{G}(k\delta^\zeta) \quad (10)$$

where the values of $\eta \approx 0.89$ and $\zeta \approx 0.29$ are obtained; $\mathcal{G}(x)$ being the universal scaling function. This implies that from the scaling analysis the estimate for the exponent γ in the limit of $\delta \rightarrow 0$ is $\gamma = \eta/\zeta \approx 3.07$. Averaging the two estimates, we quote a value of $\gamma = 2.94(14)$.

This estimated value of the degree distribution exponent close to 3, prompted us to compare this contact network with the Barabási-Albert network [20]. When a new disc is introduced, its center is selected randomly with uniform probability anywhere within the uncovered region. For an already existing disc i of radius r_i if the center of the new disc is selected within the annular ring of radius $(r_i + \delta)$ then the new disc becomes its neighbor immediately after introduction. Therefore it is more probable that a newly added disc becomes the neighbor

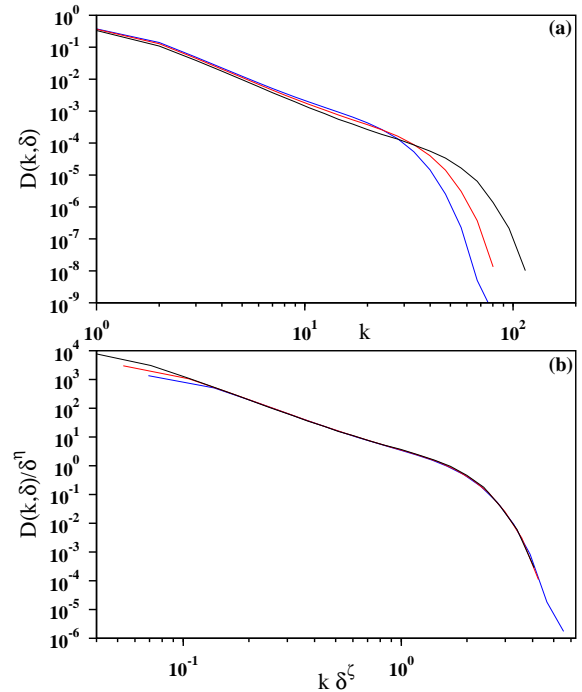


FIG. 10: (Color online) (a) The nodal degree distribution $D(k, \delta)$ of the contact network of the pattern at the percolation threshold for three different values of the growth rate $\delta = 0.0001$ (blue), 0.00004 (red) and 0.00001 (black). Binned data have been used to reduce fluctuation. (b) The finite size scaling analysis of the data in (a) where $D(k, \delta)/\delta^{0.89}$ scales as $k\delta^{0.29}$ which gives $\gamma = 3.07$.

of a larger disc than a smaller one and its probability is proportional to r_i . But this observation is until the disc i is growing, i.e., till its degree $k_i = 0$. The moment it gets its first neighbor i.e., the first link, its growth stops and it becomes frozen. Thereafter the degree k_i of i increases but its radius does not, and gradually the annular space fills up. Therefore unlike the ‘rich gets richer’ principle in the Barabási-Albert network [20] here the attachment probability is not proportional to the degree k . We studied how the average radius of all discs whose degrees are equal to k depends on k . We have plotted (not shown here) for a small value of δ using the log-log scale $\langle r(k) \rangle / \log(k)$ vs. k on a log-log scale which fit very well to a straight line indicating that the following form with logarithmic correction may be valid

$$\langle r(k) \rangle \sim k^{0.72} \log k \quad (11)$$

for the growth of the average radius of a disc with the their degree k .

5. CONCLUSION

Signature of discontinuous jumps in the Order Parameter has been observed in a continuous percolation

transition of an assembly of growing circular discs which overlap slightly before becoming frozen. Extrapolation of numerical results indicate that in the limit of the extremely slow growth rate of $\delta \rightarrow 0$ the percolation transition occurs when the area coverage is unity, i.e., the disc pattern is space-filling even at the percolation point. Surprisingly, within our numerical accuracy, it has been observed that the percolation transition of such a system has a discontinuous macroscopic jump in the Order Parameter and also associated with a vanishing width of the transition window. On the contrary, the cluster size distribution has been found to have a power law decaying functional form. We conclude that though the transition in our model is actually continuous it exhibits certain features of a discontinuous transition. We conclude that our model is yet another example like the Achlioptas Process [1], the original model of Explosive Percolation, where a similar discontinuous-like continuous transition has been observed.

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- [1] D. Achlioptas, R. M. D'Souza, and J. Spencer, *Science* **323**, 1453 (2009).
 [2] H. K. Lee, B. J. Kim and H. Park, *Phys. Rev. E* **84**, 020101(R) (2011).

- [3] S. S. Manna, *Physica A*, **391**, 2833 (2012).
 [4] R. M. Ziff, *Phys. Rev. Lett.* **103**, 045701 (2009).
 [5] R. M. Ziff, *Phys. Rev. E* **82**, 051105 (2010).
 [6] Y.S. Cho, J. S. Kim, J. Park, B. Kahng and D. Kim, *Phys. Rev. Lett.* **103**, 135702 (2009).
 [7] F. Radicchi and S. Fortunato, *Phys. Rev. Lett.* **103**, 168701 (2009).
 [8] R. K. Pan, M. Kivelä, J. Saramäki, K. Kaski, J. Kertész, *Phys. Rev. E* **83**, 046112 (2011).
 [9] O. Riordan and L. Warnke, *Ann. Appl. Prob.* **22**, 1450 (2012).
 [10] B. D. Lubachevsky and F. H. Stillinger, *J. Stat. Phys.* **60**, 561 (1990).
 [11] G. Grimmett, *Percolation*, Springer, 1999.
 [12] H. Mohammadi, E. N. Oskoe, M. Afsharchi, N. Yazdani and M. Sahimi, *Int. J. Mod. Phys. C* **20**, 1871 (2009).
 [13] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1994).
 [14] P. B. Thomas and D. Dhar, *J. Phys. A*, **27**, 2257 (1994).
 [15] S. S. Manna and H. J. Herrmann, *J. Phys. A* **24**, L481-L490 (1991).
 [16] S. S. Manna, *Physica A* **187** 373-377 (1992).
 [17] R. M. Baram and H. J. Herrmann, *Phys. Rev. Lett.* **95**, 224303 (2005).
 [18] H. J. Herrmann, G. Mantica and D. Bessis, *Phys. Rev. Lett.* **65** (1990) 3223.
 [19] J. S. Andrade, Jr., H. Herrmann, R. F. S. Andrade and L. R. da Silva, *Phys. Rev. Lett.*, **94**, 018702 (2005).
 [20] A.-L. Barabási and R. Albert, *Science*, **286**, 509 (1999).